Stochastic Coupling of Fermions

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Abstract

The stochastic quantization of the fermion field is performed starting from Dirac equations. The statistical properties of stochastic terms in Langevin equations are described by explicit formulae of a Markov process. The interaction of the field is introduced as correlation of the stochastic terms. In the long time limit free fermions disappear and proper combinations of field components propagate as a scalar boson field. The existence and uniqueness of the long time limit is proved in the first order approximation of stochastic Liouville equation.

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1 Introduction

The method of stochastic quantization [1, 2] starts from Euclidean field equations amended by addition of a random stochastic variable. Thus the change of the field during an infinitesimal increment of stochastic time t which is considered to be an auxiliary variable is described by field equation and stochastic term. The massive scalar field is described e.g. by

$$\frac{\partial}{\partial t}\phi(x,t) = (-\partial^2 + m^2)\phi(x,t) \,+\, \xi(x,t) \tag{1.1}$$

where x are coordinates in Euclidean space. (1.1) can be also regarded as Langevin equation with drift term equal to Klein-Gordon operator. The statistical properties of the fluctuation term $\xi(x,t)$ are given. One particular solution of (1.1) is of no special interest but the mean value of the correlation $\langle\phi(x,t),\phi(x',t)\rangle$ averaged over all possible fluctuations is equal in the long time limit (i.e. $t \to \infty$) to the usual result of path integral quantization of the Euclidean field theory under consideration. One has of course to prove that the long time limit exists and it is unique.

In the standard approach the fluctuation term corresponds to white noise - i.e. it is a random Gaussian variable with autocorrelation equal to

$$\langle\xi(x,t)\xi(x',t')\rangle \;=\; c\,\delta(x-x')\,\delta(t-t') \tag{1.2}$$

The finite width of the Gaussian distribution of ξ in t variable in (1.1) leads to Pauli-Villars regularization of propagators [3].

Stochastic quantization of boson fields is the easier one. Stochastic quantization of fermion fields is more elaborate and there are two approaches. The method proposed by Breit, Gupta and Zaks [4] starts also from Klein-Gordon operator as in (1.1) and the fermion propagator results from special statistical properties of the fluctuation term. In this approach

$$\frac{\partial}{\partial t}\psi(x,t) = (-\partial^2 + m^2)\psi(x,t) + \eta(x,t),$$

$$\frac{\partial}{\partial t}\bar\psi\;(x,t) = \bar\psi\;(x,t)(-\partial^2 + m^2) + \bar\eta\;(x,t)$$

and

$$\langle\eta(x,t)\bar\eta\;(x',t')\rangle \;=\; c\,(-i\partial\!\!\!/ \;+m)\,\delta(x-x')\,\delta(t-t'). \tag{1.3}$$

Thus quantization of boson and fermion fields is put on the same footing and the stochastic time has in both two cases the same dimension $[t] = m^{-2}$. This is convenient for perturbative description of interacting fermion fields.

The second method proposed by Fukai et al. [5] starts from Dirac operators

$$\frac{\partial}{\partial t}\psi(x,t) = (i\partial\!\!\!/ \;-m)\psi(x,t) + \xi(x,t),$$

$$\tag{1.4}$$

$$\frac{\partial}{\partial t}\bar\psi\;(x,t) = \bar\psi\;(x,t)(-i\partial\!\!\!/ \;-m) + \bar\xi\;(x,t)$$

and the stochastic time t has the real time dimension. It seems to be quite difficult to implement this method in the case of interacting fermion fields within the framework of

standard perturbation approach. However, it allows to quantize fermions in such a way that correlations of proper fermion field combinations are equal to autocorrelations of a boson field. Such procedure is the subject of this paper.

The idea is to introduce the interaction between fermion fields via correlation of fluctuation terms. Direct consequence of simple correlations of the type (1.2) is locality of resulting theory. As the correlations of the fluctuation terms are related to those of the field components, more elaborate probability distribution of stochastic terms can lead, in addition to ensuring the locality, also to dynamical consequences. To achieve this it is necessary to use one multivariate distribution governing behavior of all fluctuation terms in Langevin equations at given stochastic time t. The sequence of these distributions will form a Markov process. It is evident that random variables in (1.4) have to be of anticommuting nature. Markov processes of Grassmann numbers are constructed and studied in the Section 2 of the paper. Lowest order propagator based on Langevin equations with correlated fluctuation terms is calculated in the Section 3. The existence and the uniqueness of the long time limit of the studied system is proved in first order approximation in the Section 4. The Section 5 of the paper is devoted to conclusions and discussion.

2 Markov chains of Grassmann numbers

Classical Markov processes [1] are based on the notion of probability distribution. However, there is no such concept as probability measure in Grassmann number calculus. Hence one has to adopt a more general approach outlined e.g. in [8] and also used in [5]. It is possible to use as a probability distribution of Grassmann variables $\xi, \bar{\xi}$ (henceforth it will be dealt always with pairs of Grassmann numbers) a function which can be generally written as

$$P(\xi, \bar{\xi}) = c_0 + c_1\xi + c_2\bar{\xi} + c_3\xi\bar{\xi} \tag{2.1}$$

and such that the integral

$$\int f(\xi, \bar{\xi})P(\xi, \bar{\xi})\, d\xi d\bar{\xi} \;\equiv\; \langle f(\xi, \bar{\xi})\rangle \tag{2.2}$$

gives correct expectation values $\langle f(\xi, \bar{\xi})\rangle$ for any function f. It is reasonable to assume that P commutes with any Grassmann number. The conventions of Grassmann variables integration and derivations (left-derivatives) are the same as those used in [5],

$$\int d\xi = 0\,,\; \int \xi\, d\xi = i\,,\; \int d\bar{\xi} = 0\,,\; \int \bar{\xi}\, d\bar{\xi} = i \tag{2.3}$$

so that e.g.

$$\int \xi\bar{\xi}\, d\xi d\bar{\xi} = 1.$$

$$\frac{\partial}{\partial\xi}(\xi\bar{\xi}) = \bar{\xi}\;,\; \frac{\partial}{\partial\bar{\xi}}(\xi\bar{\xi}) = -\frac{\partial}{\partial\bar{\xi}}(\bar{\xi}\,\xi) = -\xi\,. \tag{2.4}$$

Classical Markov process is given by the probability distribution $P(y_1, t_1)$ of stochastic variable y_1 at certain time t_1 (with the abbreviation $y_1 \equiv y(t_1)$) and the transition

[1] An excellent introduction into Markov processes and stochastic differential equations can be found in [6].

probability $T(y_2,t_2 \mid y_1,t_1)$ which determines the probability distribution of y at any time $t_2 \geq t_1$

$$P(y_2,t_2) = \int T(y_2,t_2 \mid y_1,t_1) P(y_1,t_1)\, dy_1 \tag{2.5}$$

The transition probability satisfies Chapman-Kolmogorov equation for any $t_3 > t_2 > t_1$

$$T(y_3,t_3 \mid y_1,t_1) = \int T(y_3,t_3 \mid y_2,t_2) T(y_2,t_2 \mid y_1,t_1)\, dy_2 \tag{2.6}$$

and it is normalized

$$\int T(y_2,t_2 \mid y_1,t_1)\, dy_2 = 1. \tag{2.7}$$

One of the most simple stochastic processes one can use is the Orenstein-Uhlenbeck process. The natural step therefore will be to construct Grassmann variable analog to this process which is the unique stationary Gaussian Markov process and it is defined by

$$P(y_1) = \frac{1}{\sqrt{2\pi\sigma_E}}\; e^{-\frac{y_1^2}{2\sigma_E}}, \tag{2.8}$$

$$T(y_2,t_2 \mid y_1,t_1) = \frac{1}{\sqrt{2\pi\sigma_E(1-e^{-2M\tau})}}\; \exp\left[-\frac{(y_2 - y_1 e^{-M\tau})^2}{2\sigma_E(1-e^{-2M\tau})}\right], \tag{2.9}$$

where $\tau = t_2 - t_1$.

The multivariate distributions are needed but for simplicity in the following the functions of only one pair of Grassmann variables ξ and $\bar{\xi}$ will be used. All necessary mathematical properties can be demonstrated by means of such functions. One can take as Grassmann Gaussian probability distribution

$$P(\xi,\bar{\xi}\,) = -1 - \bar{\xi}\,\xi \tag{2.10}$$

and readily check that the normalization condition holds

$$\int P(\xi,\bar{\xi}\,)\, d\xi d\bar{\xi} = 1. \tag{2.11}$$

If one tries as transition probability distribution

$$T(\xi_2,\bar{\xi}_2 \mid \xi_1,\bar{\xi}_1) = -1 - \bar{\xi}_2\xi_2 - e^{\pm M(t_2-t_1)}[\xi_2\bar{\xi}_1 - \bar{\xi}_2\xi_1] \tag{2.12}$$

then the following equations hold

$$\int T(\xi_2,\bar{\xi}_2 \mid \xi_1,\bar{\xi}_1) P(\xi_1,\bar{\xi}_1)\, d\xi_1 d\bar{\xi}_1 = P(\xi_2,\bar{\xi}_2), \tag{2.13}$$

$$\int T(\xi_3,\bar{\xi}_3 \mid \xi_2,\bar{\xi}_2) T(\xi_2,\bar{\xi}_2 \mid \xi_1,\bar{\xi}_1)\, d\xi_2 d\bar{\xi}_2 = T(\xi_3,\bar{\xi}_3 \mid \xi_1,\bar{\xi}_1), \tag{2.14}$$

provided that $t_3 > t_2 > t_1$, and finally

$$\int T(\xi_2,\bar{\xi}_2 \mid \xi_1,\bar{\xi}_1)\, d\xi_2 d\bar{\xi}_2 = 1. \tag{2.15}$$

They correspond to (2.5,2.6) and (2.7) respectively. However, there is one significant difference - the exponent in (2.12) can be also positive in contrast to (2.9) where the positive exponent in t dependence would lead to complex T values.

The analogy of (2.10,2.12) with the classical process can be extended even further. The equation

$$\frac{\partial T}{\partial \tau} \;=\; \pm M \left[\frac{\partial}{\partial \xi_2}(\xi_2\, T) \;+\; \frac{\partial}{\partial \bar{\xi}_2}(\bar{\xi}_2\, T) \;+2\, \frac{\partial^2\, T}{\partial \bar{\xi}_2\, \partial \xi_2} \right] , \tag{2.16}$$

is the forward Kolmogorov equation for Grassmann variable Markov process (2.12), while the forward Kolmogorov equation of the classical process (2.8,2.9) reads

$$\frac{\partial T}{\partial \tau} \;=\; -M \left[\frac{\partial}{\partial y_2}(y_2\, T) \;+\; \frac{\partial^2\, T}{\partial y_2^2} \right] ,$$

The backward Kolmogorov equation for (2.12) has no analogy in corresponding classical equation. The full analogy is restored for

$$T^{'}(\xi_2, \bar{\xi}_2 \mid \xi_1, \bar{\xi}_1) \;=\; (e^{\pm 2M\tau} \;-\; 1) \;-\; (\bar{\xi}_2 \;-\; e^{\pm M\tau}\, \bar{\xi}_1)(\xi_2 \;-\; e^{\pm M\tau}\, \xi_1). \tag{2.17}$$

This transition probability distribution is the full analog to that of classical process, including both the forward and backward Kolmogorov equation. In contrast with classical Orenstein-Uhlenbeck process, which is the unique stationary Gaussian Markov process, here one has more stationary processes which can be called Gaussian. This is only natural as one deals with truncated Taylor series. The Kolmogorov equations of the process of anticommuting variables (2.17) have the same asymptotics for $\tau \to \pm\infty$ as the classical process. In the following the analog of simpler (2.12) will be used.

The stochastic variables which will enter the Langevin equations will be represented by quartets of functions which ascribe to each point in Euclidean space-time x and to each value of stochastic time t four random Grassmann numbers. To achieve the desired result - coupling of four fermions - it turns out that the probability distributions of these variables have to be functions of two pairs of such stochastic variables ($\xi_\mu(x,t)\,,\, \eta_\mu(x,t)$ and $\bar{\xi}_\mu(x',t)\,,\, \bar{\eta}_\mu(x',t)$, where $\mu = 0,1,2,3$). These pairs will be defined on a product of two Euclidean spaces $x \times x'$ and their distribution in this space will be described by δ-function $\delta(x = x')$. Further one has to suppose that these "quartets of quartets" have nonzero product of all their components - i.e. all 16 Grassmann numbers are different. Thus the probability distribution in x' becomes for any x

$$\mathcal{P}(\xi(x',t), \bar{\xi}(x',t), \eta(x',t), \bar{\eta}(x',t)) \mid_x \equiv P(\xi(x,t), \bar{\xi}(x,t), \eta(x,t), \bar{\eta}(x,t))\, \delta(x - x') \tag{2.18}$$

and the transition probability distribution as a function of y' will be

$$\begin{aligned} &\mathcal{T}(\xi(y',t_2), \bar{\xi}(y',t_2), \eta(y',t_2), \bar{\eta}(y',t_2) \,|\, \xi(x,t_1), \bar{\xi}(x,t_1), \eta(x,t_1), \bar{\eta}(x,t_1)) \mid_y \equiv \\ &T(\xi(y,t_2), \bar{\xi}(y,t_2), \eta(y,t_2), \bar{\eta}(y,t_2) \,|\, \xi(x,t_1), \bar{\xi}(x,t_1), \eta(x,t_1), \bar{\eta}(x,t_1))\, \delta(y - y') \end{aligned} \tag{2.19}$$

for any x, y. Both in (2.18) and (2.19) the greek indices are omitted. In rigorous approach the probability distributions in Euclidean space in (2.18) and (2.19) should be e.g. of the type listed in (2.8). One can consider the δ-functions in (2.18,2.19) to result from the limit $\sigma_E \to 0$ in (2.8). Simultaneously the σ_E parameter disappears. In fact one allows in (2.1) as c-coefficients not only c-functions but also c-distributions.

As the random Grassmann variable distribution $P(\xi(x,t), \bar{\xi}(x,t), \eta(x,t), \bar{\eta}(x,t))$ in (2.18) at given stochastic time t_1 the following function will be taken as ansatz

$$P(\xi^1, \bar{\xi}^1, \eta^1, \bar{\eta}^1) \;=\; \prod_{\mu=0}^{3} \eta^1_\mu \prod_{\rho=0}^{3} \bar{\xi}^1_\rho \prod_{\sigma=0}^{3} \xi^1_\sigma \prod_{\nu=0}^{3} \bar{\eta}^1_\nu \;+\; \sum_{i,j=0}^{3} (\prod_{\substack{\mu=0\\ \mu\neq i}}^{3} \eta^1_\mu \prod_{\substack{\rho=0\\ \rho\neq j}}^{3} \bar{\xi}^1_\rho \prod_{\substack{\sigma=0\\ \sigma\neq j}}^{3} \xi^1_\sigma \prod_{\substack{\nu=0\\ \nu\neq i}}^{3} \bar{\eta}^1_\nu) \tag{2.20}$$

where the greek indices of $\xi, \bar{\xi}, \eta$ and $\bar{\eta}$ on the left hand side of (2.20) are omitted and the x dependence is omitted on both sides. The stochastic time dependence is indicated by upper right index, e.g. $\eta(t_1) \equiv \eta^1$. Transition probability distribution on right hand side of (2.19) written in the same notation reads

$$
\begin{aligned}
T(\xi^2, \bar{\xi}^2, \eta^2, \bar{\eta}^2 \mid \xi^1, \bar{\xi}^1, \eta^1, \bar{\eta}^1) \; &= \; P(\xi^2, \bar{\xi}^2, \eta^2, \bar{\eta}^2) \, + \\
+ \; k \sum_{i,j=0}^{3} (-1)^{i+j} & \prod_{\substack{\mu=0 \\ \mu \neq i}}^{3} \eta^2_\mu \prod_{\substack{\rho=0 \\ \rho \neq j}}^{3} \bar{\xi}^2_\rho \prod_{\sigma=0}^{3} \xi^2_\sigma \prod_{\nu=0}^{3} \bar{\eta}^2_\nu \cdot \bar{\xi}^1_j \, \eta^1_i \; + \\
+ \; k \sum_{i,j=0}^{3} (-1)^{i+j} & \prod_{\mu=0}^{3} \eta^2_\mu \prod_{\rho=0}^{3} \bar{\xi}^2_\rho \prod_{\substack{\sigma=0 \\ \sigma \neq j}}^{3} \xi^2_\sigma \prod_{\substack{\nu=0 \\ \nu \neq i}}^{3} \bar{\eta}^2_\nu \cdot \bar{\eta}^1_i \, \xi^1_j \; ,
\end{aligned}
\tag{2.21}
$$

where

$$
k \; = \; e^{\pm M \tau} \; , \; \tau \; = \; t_2 \; - \; t_1 \; , \; t_2 \; > \; t_1 \, .
$$

Distributions (2.18) and (2.19) obey the relations (2.13) - (2.15) amended by corresponding space-time integrations preceeding the Grassmann variable integrations. This stochastic process leads to correlations

$$
\langle \bar{\xi}_\rho(x_1, t_1) \, \eta_\mu(x'_1, t_1) \, \bar{\eta}_\nu(x_2, t_2) \, \xi_\sigma(x'_2, t_2) \rangle \; = c_E \, \delta(x_1 - x'_1) \, \delta(x_2 - x'_2) \, \delta_{\mu\,\nu} \, \delta_{\rho\,\sigma} \, e^{\pm M (t_2 - t_1)}
\tag{2.22}
$$

where $t_1 \, \leq \, t_2$. The constant $c_E \; = \; 1$, $[c_E] \; = \; m^{-8}$ is introduced to adjust dimensions. As the stochastic variables will disappear after integrations in all applications in Sections 3 and 4 it is simpler to consider them to be dimensionless and to adjust dimensions in Langevin equations by another constant r_D. This constant can be also used to readjust the numerical value of c_E so that it remains equal to 1 even after change of units.

All expectation values of single Grassmann variables or incomplete pairs $\eta \, \bar{\xi}$ or $\bar{\eta} \, \xi$ are are equal to zero, e.g.

$$
\begin{aligned}
\langle \, \bar{\xi}_\mu(x_1, t_1) \, \rangle \; = \; \langle \, \eta_\mu(x_1, t_1) \, \rangle \; = \; \langle \, \bar{\eta}_\mu(x_1, t_1) \, \eta_\nu(x_1, t_1) \, \rangle \; = \\
= \; \langle \, \bar{\xi}_\mu(x_1, t_1) \, \eta_\nu(x_1, t_1) \, \bar{\eta}_\rho(x_1, t_1) \, \rangle \; = \; etc. \; = \; 0.
\end{aligned}
\tag{2.23}
$$

Also if there is not complete pair $\eta \, \bar{\xi}$ or $\bar{\eta} \, \xi$ at any time t the correlation is equal to zero, e.g.

$$
\langle \, \bar{\xi}_\rho(x_1, t_1) \, \eta_\mu(x_2, t_2) \, \bar{\eta}_\nu(x_3, t_3) \, \xi_\sigma(x_3, t_3) \, \rangle \; = \; 0
\tag{2.24}
$$

for $t_1 \, < \, t_2 \, \leq \, t_3$. The expectation values are zero also in case of the combinations like

$$
\begin{aligned}
\langle \, \bar{\xi}_\rho(x_1, t_1) \, \eta_\mu(x_1, t_1) \, \bar{\xi}_\sigma(x_2, t_2) \, \eta_\nu(x_2, t_2) \, \rangle \; = \\
= \; \langle \, \xi_\rho(x_1, t_1) \, \bar{\eta}_\mu(x_1, t_1) \, \xi_\sigma(x_2, t_2) \, \bar{\eta}_\nu(x_2, t_2) \, \rangle \; = \; 0
\end{aligned}
\tag{2.25}
$$

assuming $t_1 \, \leq \, t_2$. Also the mean value

$$
\langle \, \bar{\xi}_\rho(x_1, t_1) \, \xi_\mu(x_2, t_2) \, \bar{\eta}_\nu(x_3, t_3) \, \eta_\sigma(x_4, t_4) \, \rangle \; = \; 0
\tag{2.26}
$$

whenever the times t_1, t_2, t_3 and t_4 are not equal each other.

The higher moments are reduced to the product of the lower ones, e.g.

$$\begin{aligned}
&\langle\bar\xi_\rho(x_1,t_1)\eta_\mu(x_1,t_1)\bar\eta_\nu(x_2,t_2)\xi_\sigma(x_2,t_2)\bar\xi_\alpha(x_3,t_3)\eta_\gamma(x_3,t_3)\bar\eta_\delta(x_4,t_4)\xi_\beta(x_4,t_4)\rangle = \\
&\quad\langle\bar\eta_\rho(x_1,t_1)\xi_\mu(x_1,t_1)\bar\xi_\nu(x_2,t_2)\eta_\sigma(x_2,t_2)\bar\xi_\alpha(x_3,t_3)\eta_\gamma(x_3,t_3)\bar\eta_\delta(x_4,t_4)\xi_\beta(x_4,t_4)\rangle \qquad (2.27)\\
&= \langle\bar\eta_\rho(x_1,t_1)\xi_\mu(x_1,t_1)\bar\xi_\nu(x_2,t_2)\eta_\sigma(x_2,t_2)\rangle\langle\bar\xi_\alpha(x_3,t_3)\eta_\gamma(x_3,t_3)\bar\eta_\delta(x_4,t_4)\xi_\beta(x_4,t_4)\rangle
\end{aligned}$$

provided that $t_1 < t_2 < t_3 < t_4$.

3 Coupled fermions

The subject of the interest will be a fermion field ψ with no other interaction except the one represented by Markov process (2.18),(2.19). The Langevin equations in this case are

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_\alpha &= (i\partial\!\!\!/ - m)_{\alpha\beta}\,\psi_\beta + r_D\,(\xi_\alpha+\eta_\alpha) \\
\frac{\partial}{\partial t}\bar\psi_\alpha &= -\bar\psi_\beta\,(i\partial\!\!\!/ + m)_{\beta\alpha} + r_D\,(\bar\xi_\alpha+\bar\eta_\alpha)
\end{aligned} \qquad (3.1)$$

where

$$\partial\!\!\!/ = \gamma^\mu\,\partial_\mu = \gamma^\mu\,\frac{\partial}{\partial x^\mu} \qquad (3.2)$$

and γ^μ are Dirac matrices in Euclidean space, satisfying

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} = -\,2\,\delta_{\mu\nu}\,. \qquad (3.3)$$

The r_D is a constant with dimension $[m^{5/2}]$. The stochastic variables are dimensionless as discussed in Sec. 2.

To solve (3.1) it is useful to introduce retarded Green functions

$$\begin{aligned}
G(x,t) &= \Theta(t)\int\frac{d^4p}{(2\pi)^4}\,\exp[-(p\!\!\!/ + m)t+ipx] \\
\bar G\,(x,t) &= \Theta(t)\int\frac{d^4p}{(2\pi)^4}\,\exp[-(p\!\!\!/ + m)t-ipx]
\end{aligned} \qquad (3.4)$$

which satisfy

$$G(x,t) = \bar G\,(x,t) = 0 \quad\text{for}\quad t<0\,,$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}G_{\alpha\gamma}(x,t) - (i\partial\!\!\!/ - m)_{\alpha\beta}G_{\beta\gamma}(x,t) &= \delta_{\alpha\gamma}\,\delta^4(x)\,\delta(t) \\
\frac{\partial}{\partial t}\bar G_{\alpha\gamma}(x,t) - \bar G_{\alpha\beta}(x,t)(-i\partial\!\!\!/ - m)_{\beta\gamma} &= \delta_{\alpha\gamma}\,\delta^4(x)\,\delta(t)\,.
\end{aligned} \qquad (3.5)$$

Solutions to (3.1) can be written as convolutions

$$\psi_\alpha(x,t) = r_D\int_0^t d\tau\int d^4x'\,G_{\alpha\rho}(x-x',t-\tau)(\xi_\rho(x',\tau)+\eta_\rho(x',\tau))$$

$$\bar{\psi}_\alpha(x,t) = r_{_D} \int_0^t d\tau \int d^4x' \, (\bar{\xi}_\rho(x',\tau) + \bar{\eta}_\rho(x',\tau))\bar{G}_{\rho\alpha}(x-x',t-\tau) \tag{3.6}$$

The nature of fluctuation terms in (3.1) implies that the simplest nonzero correlation function one can calculate (see(2.22)) is

$$\begin{aligned}
\langle\bar{\psi}_\alpha(x_1,t_1)\psi^\alpha(x_2,t_2)\bar{\psi}_\beta(y_1,t_3)\psi^\beta(y_2,t_4)\rangle \;&=\; r_{_D}^4\int_0^{t_1}\!d\tau_1\int_0^{t_2}\!d\tau_2\int_0^{t_3}\!d\tau_3\int_0^{t_4}\!d\tau_4\int d^4x_1'\int d^4x_2'\int d^4y_1'\int d^4y_2' \\
&\langle((\bar{\xi}_\sigma(x_1',\tau_1)+\bar{\eta}_\sigma(x_1',\tau_1))\bar{G}_{\sigma\alpha}(x_1-x_1',t_1-\tau_1)G_{\alpha\rho}(x_2-x_2',t_2-\tau_2)(\xi_\rho(x_2',\tau_2)+\eta_\rho(x_2',\tau_2)) \\
&(\bar{\xi}_\mu(y_1',\tau_3)+\bar{\eta}_\mu(y_1',\tau_3))\bar{G}_{\mu\beta}(y_1-y_1',t_3-\tau_3)G_{\beta\nu}(y_2-y_2',t_4-\tau_4)(\xi_\nu(y_2',\tau_4)+\eta_\nu(y_2',\tau_4))\rangle \\
= \;& r_{_D}^4\int_0^{t_1}\!d\tau_1\int_0^{t_2}\!d\tau_2\int_0^{t_3}\!d\tau_3\int_0^{t_4}\!d\tau_4\int d^4x_1'\int d^4x_2'\int d^4y_1'\int d^4y_2' \qquad (3.7)\\
&\bar{G}_{\sigma\alpha}(x_1-x'_1,t_1-\tau_1)G_{\alpha\rho}(x_2-x'_2,t_2-\tau_2)\bar{G}_{\mu\beta}(y_1-y'_1,t_3-\tau_3)G_{\beta\nu}(y_2-y'_2,t_4-\tau_4) \\
&\langle((\bar{\xi}_\sigma(x_1',\tau_1)+\bar{\eta}_\sigma(x_1',\tau_1))(\xi_\rho(x_2',\tau_2)+\eta_\rho(x_2',\tau_2)) \times \\
&\qquad\qquad \times(\bar{\xi}_\mu(y_1',\tau_3)+\bar{\eta}_\mu(y_1',\tau_3))(\xi_\nu(y_2',\tau_4)+\eta_\nu(y_2',\tau_4))\rangle
\end{aligned}$$

where $\langle ...\rangle$ represents integration over Grassmann variables. Due to the properties of stochastic variables (2.25,2.26), integral (3.7) over region $t_1 \times t_2 \times t_3 \times t_4$ is equal to zero. To stay on the manifold where the mean value $\langle ...\rangle \neq 0$, it is necessary to impose condition that the time integrations are carried out only for $t_1 = t_2 \leq t_3 = t_4$. Hence (3.7) will be first integrated with the weight

$$c_\tau \, \delta(\tau_1-\tau_2)\,\delta(\tau_3-\tau_4)\,, \tag{3.8}$$

where $c_\tau = 1$, $[c_\tau] = m^{-2}$. The numerical value of c_τ can be again adjusted to 1 by means of $r_{_D}$ constant as in all expressions leading to measurable quantities one has always the product $r_{_D}^4\, c_\tau\, c_E$ which is dimensionless. The condition (3.8) means, that the product $\bar{\psi}_\alpha\psi^\alpha$ will be always taken in the same stochastic time instant. From (3.7) one arrives at

$$\begin{aligned}
\langle\bar{\psi}_\alpha\psi^\alpha\bar{\psi}_\beta\psi^\beta\rangle \;&=\; 2c_\tau c_E r_{_D}^4\int_0^\infty\!d\tau_1\int_0^\infty\!d\tau_3\int d^4x'_1\int d^4y'_1\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_4}{(2\pi)^4} \\
&\Theta\,(t_1-\tau_1)\exp[-(\not{p}_1+m)(t_1-\tau_1)-ip_1x_1]_{\sigma\alpha} \\
&\Theta\,(t_2-\tau_1)\exp[-(\not{p}_2+m)(t_2-\tau_1)+ip_2x_2]_{\alpha\rho}\exp[i(p_1-p_2)x'_1] \\
&\Theta\,(t_3-\tau_3)\exp[-(\not{p}_3+m)(t_3-\tau_3)-ip_3y_1]_{\mu\beta} \\
&\Theta\,(t_4-\tau_3)\exp[-(\not{p}_4+m)(t_4-\tau_3)+ip_4y_2]_{\beta\nu}\exp[i(p_3-p_4)y'_1] \\
&\delta_{\rho\mu}\;\delta_{\sigma\nu}\;\exp[\pm M\mid\tau_3-\tau_1\mid]
\end{aligned} \tag{3.9}$$

carrying out also the integrations over δ-functions

$$\delta(x'_1-x'_2)\,\delta(y'_1-y'_2)$$

which come from (2.22). Further integrations lead to

$$\begin{aligned}
\langle\bar{\psi}_\alpha\psi^\alpha\bar{\psi}_\beta\psi^\beta\rangle \;=\; & 2r\int_0^\infty\!d\tau_1\int_0^\infty\!d\tau_3\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\;\Theta(T_1-\tau_1)\Theta(T_3-\tau_3) \\
&\exp[-2(\not{p}_1+m)(T_1-\tau_1)]_{\sigma\rho}\exp[-2(\not{p}_3+m)(T_3-\tau_3)]_{\rho\sigma} \\
&\exp[i(x_2-x_1)p_1]\;\exp[i(y_2-y_1)p_3]\;\exp[\pm M\mid\tau_3-\tau_1\mid]
\end{aligned} \tag{3.10}$$

where $r = r_D^4\, c_\tau\, c_E$ and it is dimensionless. Further

$$T_1 = min(t_1, t_2)\,,\, T_3 = min(t_3, t_4)\,.$$

So far only δ-functions have been integrated. To integrate over remaining stochastic time variables, one needs to prove convergence of the integrals (3.10). This is easily done realizing that there exists a unitary transformation which diagonalizes the matrix $(\not p + m)$ [5].

$$(\not p + m) = \mathcal{U}^{-1}(p) \begin{pmatrix} i\sqrt{p^2}+m & 0 & 0 & 0 \\ 0 & i\sqrt{p^2}+m & 0 & 0 \\ 0 & 0 & -i\sqrt{p^2}+m & 0 \\ 0 & 0 & 0 & -i\sqrt{p^2}+m \end{pmatrix} \mathcal{U}(p)\,. \quad (3.11)$$

After integrations over τ_1 and τ_3, where it is necessary to integrate separately and then to add contributions from the regions $\tau_3 > \tau_1$ and $\tau_1 > \tau_3$, one arrives at

$$\begin{aligned} \langle \bar\psi_\alpha \psi^\alpha \bar\psi_\beta \psi^\beta \rangle = 2r \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4}\; \exp[i(x_2 - x_1)p_1]\, \exp[i(y_2 - y_1)p_3] \;\times \\ \times\, Tr \Bigg\{ \exp[-(2\not p_1 + 2\not p_3 + 4m)T] \\ \left(\frac{\exp[(2\not p_1 + 2\not p_3 + 4m)T] - 1}{(2\not p_1 + 2\not p_3 + 4m)(2\not p_3 + 2m \mp M)} - \frac{\exp[(2\not p_1 + 2m \pm M)T] - 1}{(2\not p_3 + 2m \mp M)(2\not p_1 + 2m \pm M)} + \right. \\ \left. + \frac{\exp[(2\not p_1 + 2\not p_3 + 4m)T] - 1}{(2\not p_1 + 2\not p_3 + 4m)(2\not p_1 + 2m \mp M)} - \frac{\exp[(2\not p_3 + 2m \pm M)T] - 1}{(2\not p_3 + 2m \pm M)(2\not p_1 + 2m \mp M)} \right) \Bigg\}\,, \end{aligned}$$
(3.12)

where it has been put $T \equiv T_1 = T_3$ in accordance with standard stochastic quantization procedure [2]. To carry out the limit $T \to \infty$ one has again to diagonalize the exponential operators by the unitary transformation (3.11). The result is

$$\begin{aligned} \langle \bar\psi_\alpha \psi^\alpha \bar\psi_\beta \psi^\beta \rangle = 2r \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4}\; \exp[i(x_2 - x_1)p_1]\, \exp[i(y_2 - y_1)p_3] \\ Tr \left\{ \frac{1}{\not p_1 + \not p_3 + 2m} \left(\frac{1}{2\not p_3 + 2m \mp M} + \frac{1}{2\not p_1 + 2m \mp M} \right) \right\}\,, \end{aligned} \quad (3.13)$$

where the signs $\mp M$ correspond to $e^{\pm M\tau}$ in (2.22). In the case of correlations with $e^{+M\tau}$ it is necessary to assume

$$2\,m > M \quad (3.14)$$

to assure convergence of (3.12) when $T \to +\infty$.

The result (3.13) is a four-point function and it is desirable to reduce it to a two-point function. To achieve this, new momenta have to be introduced

$$p^* = p_1 + p_3\;,\; p = p_1 - p_3 \quad (3.15)$$

which substituted to (3.13) transform it to

$$\begin{aligned} \langle \bar\psi_\alpha \psi^\alpha \bar\psi_\beta \psi^\beta \rangle = 2r \int \frac{d^4p^*}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \quad (3.16) \\ \exp\left[ip^*[(x_2 - x_1) + (y_2 - y_1)]\right] \exp[ip[(x_2 - x_1) - (y_2 - y_1)]] \\ Tr \left\{ \frac{1}{\not p^* + 2m} \left(\frac{1}{\not p^* - \not p + 2m \mp M} + \frac{1}{\not p^* + \not p + 2m \mp M} \right) \right\}\,. \end{aligned}$$

The space coordinate dependence splits into two parts. In one exponent one has the sum of relative coordinates of the fields $\bar{\psi}$ and ψ - this corresponds to p^*. In the other exponent the "relative coordinate of relative coordinates" corresponds to p. It is useful to introduce also a new set of Euclidean space variables

$$
\begin{aligned}
z &= [(x_2 - x_1) - (y_2 - y_1)] \\
u &= [(x_2 - x_1) + (y_2 - y_1)] \\
v &= [\, y_1 + y_2 \,] \\
w &= [\, x_1 + x_2 \,]
\end{aligned}
\tag{3.17}
$$

The expression (3.16) depends on z and u only. Variables v and w are chosen to complete the set of linearly independent variables. Alternative choices are

$$
v' = [\, x_2 + x_1 + y_2 + y_1 \,] \; , \; w' = w
$$

or

$$
v'' = v' \quad , \quad w'' = [\, y_1 + y_2 \,] \, .
$$

One possible way, how to reduce the number of space variables, is to integrate (3.16) over u. This yields

$$
\begin{aligned}
\langle\langle \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \rangle\rangle \equiv \int du \, \langle \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \rangle \; = \; 2r \int d^4 p^* \int \frac{d^4 p}{(2\pi)^4} \, \delta(p^*) \, e^{ipz} \\
Tr \left\{ \frac{1}{p^* + 2m} \left(\frac{1}{p^* - p + 2m \mp M} + \frac{1}{p^* + p + 2m \mp M} \right) \right\} .
\end{aligned}
\tag{3.18}
$$

and subsequent integration over p^* leads to

$$
\langle\langle \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \rangle\rangle \; = \; 2r \int \frac{d^4 p}{(2\pi)^4} \, e^{ipz} \left(\frac{2m \mp M}{m} \right) \frac{1}{p^2 + (2m \mp M)^2} \, .
\tag{3.19}
$$

The further reductions of space variables imposed by conditions $v = w = 0$ lead to

$$
\langle\langle \bar{\psi}_\alpha(-x) \psi^\alpha(x) \bar{\psi}_\beta(-y) \psi^\beta(y) \rangle\rangle \; = \; N \int \frac{d^4 p}{(2\pi)^4} \, e^{ip(x-y)} \, \frac{1}{p^2 + (2m \mp M)^2} \, ,
\tag{3.20}
$$

where all coefficients were absorbed into the constant N. By means of the same procedure applied to the conjugated field combinations one gets

$$
\langle\langle \bar{\psi}_\alpha(x) \psi^\alpha(-x) \bar{\psi}_\beta(y) \psi^\beta(-y) \rangle\rangle \; = \; N \int \frac{d^4 p}{(2\pi)^4} \, e^{-ip(x-y)} \, \frac{1}{p^2 + (2m \mp M)^2} \, ,
\tag{3.21}
$$

which is in accordance with the condition that the overall center of gravity is at rest.

4 Long time limit

In the previous section an expectation value and its limit when the stochastic time goes to infinity was calculated. To be able to ascribe some meaning to this result, one has to know that it is based on proper long time limit behavior of the whole process. This means

- one has to prove that the distribution of the solutions to the equations (3.1) has a limit when the stochastic time goes to infinity and this limit is unique.

To achieve this goal the Fokker-Planck equation cannot be used, as it is derived under assumption that the relaxation time of fluctuations is very small compared to observation time. The considered stochastic process of Grassmann numbers allows both signs in the auto-correlation function ($e^{\pm M\tau}$) and thus it is desirable to keep full control of the time evolution rather then to use an approximation. A significant virtue of stochastic quantization is that it offers the whole arsenal of stochastic differential equations. It is possible to use the stochastic Liouville equation [6, 7]. The idea is to use the well known hydrodynamic equation $\dot{\rho} = -div\,\rho\,\vec{v}$ in the space of solutions to (3.1). The ρ becomes then the density of solutions ψ and $\bar{\psi}$ and there exists a theorem [6, 7] saying that $\langle\rho(\psi,\bar{\psi},t)\rangle$, i.e. the solution density averaged over all fluctuations $\eta,\xi,\bar{\eta},\bar{\xi}$ is the probability distribution of $\psi,\bar{\psi}$ at stochastic time t. It is possible to prove the existence and uniqueness of the long time limit from the properties of such probability distribution. The Liouville equation in this case reads

$$\frac{\partial}{\partial t}\,\rho(\psi,\bar{\psi},t) = +\frac{\partial}{\partial\,\psi_\nu}\,[\dot{\psi}_\nu\,\rho(\psi,\bar{\psi},t)] + \frac{\partial}{\partial\,\bar{\psi}_\nu}\,[\dot{\bar{\psi}}_\nu\,\rho(\psi,\bar{\psi},t)], \tag{4.1}$$

where the plus sign on the right hand side comes from derivation of the Liouville equation with anticommuting variables [5].

Substitution of the right hand sides of (3.1) into (4.1) yields in p-representation

$$\begin{aligned}\frac{\partial\rho}{\partial t} = \Big\{&-8m + (\not{p}+m)_{\nu\mu}\psi_\mu\frac{\partial}{\partial\psi_\nu} + (\not{q}+m)_{\nu\mu}\bar{\psi}_\nu\frac{\partial}{\partial\bar{\psi}_\mu}\\ &-r_D\,(\xi\prime_\nu+\eta\prime_\nu)\frac{\partial}{\partial\psi_\nu} - r_D\,(\bar{\xi}\prime_\nu+\bar{\eta}\prime_\nu)\frac{\partial}{\partial\bar{\psi}_\nu}\Big\}\,\rho(\psi,\bar{\psi},t)\end{aligned} \tag{4.2}$$

where $\xi\prime,\bar{\xi}\prime,\eta\prime$ and $\bar{\eta}\prime$ are the Fourier pictures of stochastic terms and they have the same correlations as in (2.22) -(2.28). The probability distributions (2.18,2.19) have also to be transformed, e.g. the (2.18) transforms as follows

$$\begin{aligned}&\mathcal{P}\,(\xi(p,t),\bar{\xi}(p',t),\eta(p,t),\bar{\eta}(p',t)) = \mathcal{F}[\mathcal{P}(\xi(x,t),\bar{\xi}(x',t),\eta(x,t),\bar{\eta}(x',t))]\\ &= \int\frac{d^4x}{(2\pi)^4}\int\frac{d^4x'}{(2\pi)^4}\;e^{ipx}\,e^{-ip'x'}\;\;P(\xi(x,t),\bar{\xi}(x,t),\eta(x,t),\bar{\eta}(x,t))\,\delta(x-x')\\ &= \frac{1}{(2\pi)^4}\int\frac{d^4x}{(2\pi)^4}\;e^{i(p-p')x}\;\;P(\xi(x,t),\bar{\xi}(x,t),\eta(x,t),\bar{\eta}(x,t))\end{aligned} \tag{4.3}$$

In order to solve (4.2) it is necessary to transform it to the interaction representation. The stochastic time independent part of the operator on the right hand side of (4.2) can be denoted as

$$A_0 \;=\; -8m + (\not{p}+m)_{\nu\mu}\psi_\mu\frac{\partial}{\partial\psi_\nu} + (\not{q}+m)_{\nu\mu}\bar{\psi}_\nu\frac{\partial}{\partial\bar{\psi}_\mu} \tag{4.4}$$

and the rapidly varying part as

$$A_1(t) \;=\; -r_D\,(\xi\prime_\nu+\eta\prime_\nu)\frac{\partial}{\partial\psi_\nu} - r_D\,(\bar{\xi}\prime_\nu+\bar{\eta}\prime_\nu)\frac{\partial}{\partial\bar{\psi}_\nu} \tag{4.5}$$

After transformation

$$\sigma(t) \;=\; e^{-tA_0}\rho(t) \;\;,\;\; \sigma(0) \;=\; \rho(0) \tag{4.6}$$

the equation (4.2) turns into

$$\dot{\sigma}\;(t) \;=\; e^{-tA_0}A_1(t)e^{tA_0}\;\sigma(t) \;=\; -V(t)\;\sigma(t) \tag{4.7}$$

where

$$V(t) \;=\; e^{-tA_0}\;A_1(t)\;e^{tA_0}\,. \tag{4.8}$$

Solution to (4.7) can be written as the time ordered exponential

$$\sigma(t) \;=\; \lceil \exp\{-\int_0^t V(t')\;dt'\}\rceil a \tag{4.9}$$

which can be expanded as

$$\begin{aligned}\sigma(t) = a - &\int_0^t dt_1 V(t_1)a + \int_0^t dt_1\int_0^{t_1} dt_2 V(t_1)V(t_2)a - \int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 V(t_1)V(t_2)V(t_3)a \\ &+ \int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\int_0^{t_3} dt_4 V(t_1)V(t_2)V(t_3)V(t_4)a - ...\end{aligned} \tag{4.10}$$

One gets rid of all the terms in (4.10) except of those of the order $4n(n = 0, 1, 2, ..)$ by calculation of the average $\langle\sigma(t)\rangle$ over Grassmann stochastic variables. The remaining terms are nonzero provided that they have been averaged in the same way as in (3.7) with the weight (3.8). The first two nonzero terms are

$$\langle S_0\rangle \;=\; \langle a\rangle \;=\; \langle \rho(t=0)\rangle \;\equiv\; 1 \tag{4.11}$$

which is in fact normalization of probability distribution and

$$S_1 \;=\; \int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \int_0^t dt_4 V(t_1)V(t_2)V(t_3)V(t_4)a. \tag{4.12}$$

256 terms of the product in S_1 are reduced by integration over stochastic terms to the expression

$$\begin{aligned}\langle S_1\rangle = 4r &\int_0^t dt_1 e^{-t_1A_0}\frac{\partial}{\partial\psi_\nu(p_1)}\frac{\partial}{\partial\bar{\psi}_\mu(p_1)}e^{t_1A_0}\int_0^t dt_3 e^{-t_3A_0}\frac{\partial}{\partial\psi_\mu(q_3)}\frac{\partial}{\partial\bar{\psi}_\nu(q_3)}e^{t_3A_0}e^{\pm M|t_1-t_3|} \\ +4r &\int_0^t dt_1 e^{-t_1A_0}\frac{\partial}{\partial\bar{\psi}_\nu(q_1)}\frac{\partial}{\partial\psi_\mu(q_1)}e^{t_1A_0}\int_0^t dt_3 e^{-t_3A_0}\frac{\partial}{\partial\bar{\psi}_\mu(p_3)}\frac{\partial}{\partial\psi_\nu(p_3)}e^{t_3A_0}e^{\pm M|t_1-t_3|}\end{aligned} \tag{4.13}$$

where the time integrations have been carried out with the weight (3.8) and $r = r_D^4\, c_\tau\, c_E$ is dimensionless as in the (3.10). The absolute values of time difference in (4.13) come from the fact that although the time sequence is ordered, the Markovian process can run in both directions. To cope further with (4.13) it is necessary to commute the exponentials with the derivatives. This leads to

$$\begin{aligned}\langle S_1\rangle =\;\; & 4r\int_0^t dt_1\int_0^{t_1} dt_3\;\; \frac{\partial}{\partial\psi_\nu}\exp[t_1(\not{p}_1 + m \pm \frac{M}{2})]_{\nu\tau}\exp[t_1(\not{p}_1 + m \pm \frac{M}{2})]_{\omega\mu}\frac{\partial}{\partial\bar{\psi}_\mu} \\ & \qquad\qquad\qquad\quad \frac{\partial}{\partial\psi_\rho}\exp[t_3(\not{q}_3 + m \mp \frac{M}{2})]_{\rho\omega}\exp[t_3(\not{q}_3 + m \mp \frac{M}{2})]_{\tau\sigma}\frac{\partial}{\partial\bar{\psi}_\sigma} \\ +\;\; & 4r\int_0^t dt_3\int_0^{t_3} dt_1\;\; \frac{\partial}{\partial\psi_\nu}\exp[t_1(\not{p}_1 + m \mp \frac{M}{2})]_{\nu\tau}\exp[t_1(\not{p}_1 + m \mp \frac{M}{2})]_{\omega\mu}\frac{\partial}{\partial\bar{\psi}_\mu} \\ & \qquad\qquad\qquad\quad \frac{\partial}{\partial\psi_\rho}\exp[t_3(\not{q}_3 + m \pm \frac{M}{2})]_{\rho\omega}\exp[t_3(\not{q}_3 + m \pm \frac{M}{2})]_{\tau\sigma}\frac{\partial}{\partial\bar{\psi}_\sigma} \\ +\;\; & cc\,(p \leftrightarrow q)\end{aligned} \tag{4.14}$$

where one has to integrate separately and add contributions from the two possible time orderings $t_1 > t_3$ and $t_3 > t_1$. The cc part originates from the second term in (4.13).

It is now necessary to calculate an integral of the type

$$\mathcal{I}^{\beta\delta}_{\alpha\gamma} \;=\; \int_0^t d\tau (e^{A\tau})_{\alpha\beta}\,(e^{B\tau})_{\gamma\delta}. \tag{4.15}$$

By integration per partes one gets

$$\int_0^t d\tau (e^{A\tau})_{\alpha\beta}\,(e^{B\tau})_{\gamma\delta} \;=\; (e^{A\tau})_{\alpha\mu}\,A^{-1}_{\mu\beta}\,(e^{B\tau})_{\gamma\delta}|_0^t \;-\; \int_0^t d\tau (e^{A\tau})_{\alpha\mu}\,A^{-1}_{\mu\beta}\,(e^{B\tau})_{\gamma\nu}\,B_{\nu\delta} \tag{4.16}$$

and provided that $[A, B] \;=\; 0$

$$\int_0^t d\tau (e^{A\tau})_{\alpha\mu}\,(e^{B\tau})_{\gamma\nu}\,[\delta_{\mu\beta}\,\delta_{\nu\delta} \;+\; A^{-1}_{\mu\beta}\,B_{\nu\delta}] \;=\; (e^{A\tau})_{\alpha\mu}\,A^{-1}_{\mu\beta}\,(e^{B\tau})_{\gamma\delta}\;|_0^t \tag{4.17}$$

which results in

$$\int_0^t d\tau (e^{A\tau})_{\alpha\beta}\,(e^{B\tau})_{\gamma\delta} \;=\; [\,A_{\alpha\mu}\,\delta_{\gamma\nu} \;+\; \delta_{\alpha\mu}\,B_{\gamma\nu}\,]^{-1}\;(e^{A\tau})_{\mu\beta}\,(e^{B\tau})_{\nu\delta}\;\;|_0^t \tag{4.18}$$

In order to calculate (4.14) one has to substitute $A \;=\; B \;=\; \not{p}_1 + m \pm \frac{M}{2}$ (resp. $A \;=\; B \;=\; \not{q}_3 + m \mp \frac{M}{2}$). At this moment it is useful put $\not{p}_1 \;=\; (\not{p}^* + \not{p})/2$ and $\not{q}_3 \;=\; (\not{p}^* - \not{p})/2$ and to apply the procedure outlined in (3.15 - 3.19). This is equivalent to substitution $\not{p}_1 \;=\; -\not{q}_3 \;\equiv\; \not{p}/2$. Application of (4.18) to (4.14) then leads to

$$\begin{aligned}
\langle S_1\rangle \;=\; & 2r\int_0^t dt_1\,\frac{\partial}{\partial\psi_\nu}\exp[t_1(\frac{\not{p}}{2} + m \pm \frac{M}{2})]_{\nu\tau}\exp[t_1(\frac{\not{p}}{2} + m \pm \frac{M}{2})]_{\omega\mu}\frac{\partial}{\partial\bar{\psi}_\mu} \\
& \frac{\partial}{\partial\psi_\rho}\Big\{\exp[t_1(-\frac{\not{p}}{2} + m \mp \frac{M}{2})]_{\rho\gamma}\exp[t_1(-\frac{\not{p}}{2} + m \mp \frac{M}{2})]_{\tau\alpha} \\
& [-\not{p}_{\gamma\omega}\delta_{\alpha\sigma} \;-\; \delta_{\gamma\omega}\not{p}_{\alpha\sigma} \;+\; 2(2m \mp M)\,\delta_{\gamma\omega}\delta_{\alpha\sigma}]^{-1} \;+\; \mathcal{O}^{\sigma\omega}_{\tau\rho}(\exp(0))\Big\}\frac{\partial}{\partial\bar{\psi}_\sigma} \\
+\; & 2r\int_0^t dt_3\,\frac{\partial}{\partial\psi_\nu}\Big\{[\not{p}_{\nu\tau}\delta_{\gamma\omega} \;+\; \delta_{\nu\tau}\not{p}_{\gamma\omega} \;+\; 2(2m \mp M)\,\delta_{\nu\tau}\delta_{\gamma\omega}]^{-1} \\
& \exp[t_3(\frac{\not{p}}{2} + m \mp \frac{M}{2})]_{\tau\alpha}\exp[t_3(\frac{\not{p}}{2} + m \mp \frac{M}{2})]_{\omega\mu} \;+\; \mathcal{O}^{\alpha\mu}_{\nu\gamma}(\exp(0))\Big\}\frac{\partial}{\partial\bar{\psi}_\mu} \\
& \frac{\partial}{\partial\psi_\rho}\exp[t_3(-\frac{\not{p}}{2} + m \pm \frac{M}{2})]_{\rho\gamma}\exp[t_3(-\frac{\not{p}}{2} + m \pm \frac{M}{2})]_{\alpha\sigma}\frac{\partial}{\partial\bar{\psi}_\sigma} \\
+\; & cc\,(p \leftrightarrow q)
\end{aligned} \tag{4.19}$$

The terms $\mathcal{O}(\exp(0))$ come from the lower limits of the integrals. It is only their behavior as functions of the stochastic time which is relevant for further calculations. Multiplication of matrices and subsequent time integration lead to

$$\begin{aligned}
\langle S_1\rangle \;=\; & r\,e^{4mt}\,\frac{\partial}{\partial\psi_\nu}\frac{\partial}{\partial\bar{\psi}_\mu}\frac{[\not{p}_{\nu\sigma}\not{p}_{\rho\mu} + [p^2 + 2(m \mp \frac{M}{2})^2]\delta_{\nu\sigma}\delta_{\rho\mu}]}{m\,(2m \mp M)\;[p^2 + (m \mp \frac{M}{2})^2]}\frac{\partial}{\partial\psi_\rho}\frac{\partial}{\partial\bar{\psi}_\sigma} \\
+\; & \mathcal{O}^{\mu\sigma}_{\nu\rho}(\exp[t(2\not{p} + 2m \pm M)])\frac{\partial}{\partial\psi_\nu}\frac{\partial}{\partial\bar{\psi}_\mu}\frac{\partial}{\partial\psi_\rho}\frac{\partial}{\partial\bar{\psi}_\sigma} \\
+\; & \mathcal{O}^{\mu\sigma}_{\nu\rho}(\exp[t(-2\not{p} + 2m \pm M)])\frac{\partial}{\partial\psi_\nu}\frac{\partial}{\partial\bar{\psi}_\mu}\frac{\partial}{\partial\psi_\rho}\frac{\partial}{\partial\bar{\psi}_\sigma}.
\end{aligned} \tag{4.20}$$

This expression can be written in the form of the integral

$$\langle S_1 \rangle \;= 4m \int_0^t d\tau \widetilde{W}^{\mu\sigma}_{\nu\rho}(\tau) \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \bar{\psi}_\mu} \frac{\partial}{\partial \psi_\rho} \frac{\partial}{\partial \bar{\psi}_\sigma} \tag{4.21}$$

where

$$\begin{aligned} \widetilde{W}^{\mu\sigma}_{\nu\rho}(\tau) = \quad & r\, e^{4m\tau}\, \frac{[\not{p}_{\nu\sigma}\not{p}_{\rho\mu} + [p^2 + 2(m \mp \frac{M}{2})^2]\delta_{\nu\sigma}\delta_{\rho\mu}]}{m\,(2m \mp M)\; [p^2 + (m \mp \frac{M}{2})^2]} \\ & + \mathcal{O}^{\mu\sigma}_{\nu\rho}([-2\not{p} + 2m \pm M] \exp[\tau(-2\not{p} + 2m \pm M)]) \\ & + \;\; \mathcal{O}^{\mu\sigma}_{\nu\rho}([2\not{p} + 2m \pm M] \exp[\tau(2\not{p} + 2m \pm M)]). \end{aligned} \tag{4.22}$$

The equation

$$\langle \sigma \rangle = 1 \;+\; 4m \int_0^t d\tau \widetilde{W}^{\mu\sigma}_{\nu\rho}(\tau) \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \bar{\psi}_\mu} \frac{\partial}{\partial \psi_\rho} \frac{\partial}{\partial \bar{\psi}_\sigma} \tag{4.23}$$

can be then regarded as truncated expansion of the solution to the equation

$$\langle \dot{\sigma} \rangle \;=\; 4m\; \widetilde{W}^{\mu\sigma}_{\nu\rho}(t) \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \bar{\psi}_\mu} \frac{\partial}{\partial \psi_\rho} \frac{\partial}{\partial \bar{\psi}_\sigma} \langle \sigma \rangle. \tag{4.24}$$

After transformation back from the interaction representation, the equation for $\langle \rho \rangle$ becomes

$$\langle \dot{\rho} \rangle \;=\; A_0 \langle \rho(t) \rangle \;+\; 4m\; e^{tA_0} \widetilde{W}^{\mu\sigma}_{\nu\rho}(t) \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \bar{\psi}_\mu} \frac{\partial}{\partial \psi_\rho} \frac{\partial}{\partial \bar{\psi}_\sigma}\; e^{-tA_0}\; \langle \rho(t) \rangle \tag{4.25}$$

and after commutation of the field derivatives with the operator e^{-tA_0} one obtains

$$\langle \dot{\rho} \rangle \;=\; A_0 \langle \rho(t) \rangle \;+\; 4m\; W^{\mu\sigma}_{\nu\rho}(t) \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \bar{\psi}_\mu} \frac{\partial}{\partial \psi_\rho} \frac{\partial}{\partial \bar{\psi}_\sigma}\; \langle \rho(t) \rangle \tag{4.26}$$

where

$$W^{\mu\sigma}_{\nu\rho}(t) \;=\; r\; \frac{[\not{p}_{\nu\sigma}\not{p}_{\rho\mu} + [p^2 + 2(m \mp \frac{M}{2})^2]\delta_{\nu\sigma}\delta_{\rho\mu}]}{m\,(2m \mp M)\; [p^2 + (m \mp \frac{M}{2})^2]} \;+\; \mathcal{O}^{\mu\sigma}_{\nu\rho}(\exp[t(-2m \pm M)]). \tag{4.27}$$

To investigate the asymptotic properties of solutions to (4.26) it is necessary to find its eigenstates and corresponding eigenvalues. To this purpose it is useful to introduce

$$\begin{aligned} \mathbf{a}^\dagger_\nu \;&=\; \mathcal{U}_{\nu\alpha}\; \psi_\alpha \qquad , \qquad \mathbf{a}^\rho \;=\; \frac{\partial}{\partial \psi_\beta}\; \mathcal{U}^{-1}_{\beta\rho} \\ \mathbf{b}^{\dagger\mu} \;&=\; \bar{\psi}_\gamma\; \mathcal{U}^{-1}_{\gamma\mu} \qquad , \qquad \mathbf{b}_\sigma \;=\; \mathcal{U}_{\sigma\delta}\; \frac{\partial}{\partial \bar{\psi}_\delta} \end{aligned} \tag{4.28}$$

where $\mathcal{U}$ is the unitary transformation which diagonalizes the matrix $\not{p}$ and hence also W. The operators (4.28) are anticommuting

$$\{\mathbf{a}^\rho\,,\, \mathbf{a}^\dagger_\nu\} \;=\; \delta^\rho_\nu \qquad , \qquad \{\mathbf{b}_\sigma\,,\, \mathbf{b}^{\dagger\mu}\;\} \;=\; \delta^\mu_\sigma. \tag{4.29}$$

To convert (4.26) into algebraic equation one has to introduce further the states

$$\mathbf{a}^\nu \mid 0\rangle \;=\; 0 \;\;,\;\; \langle 0 \mid \mathbf{a}^\dagger_\nu \;=\; 0 \;\;,\;\; \mathbf{b}_\mu \mid 0\rangle \;=\; 0 \;\;,\;\; \langle 0 \mid \mathbf{b}^{\dagger\mu} \;=\; 0. \qquad (4.30)$$

The search for the eigenstates of the operator on the right hand side of (4.26) leads to the equation

$$0 \;=\; -8m + \mathcal{M}^\mu_\nu \mathbf{a}^\dagger_\nu \mathbf{a}^\mu + \mathcal{N}^\mu_\nu \mathbf{b}^{\dagger\nu}\mathbf{b}_\mu + 4m\, \mathcal{W}^{\mu\sigma}_{\nu\rho}\mathbf{a}^\nu \mathbf{b}_\mu \mathbf{a}^\rho \mathbf{b}_\sigma \;\equiv\; \mathbf{R} \qquad (4.31)$$

where

$$\mathcal{M}^\mu_\nu \;=\; \mathcal{U}_{\nu\alpha}(\not{p} + m)_{\alpha\beta}\mathcal{U}^{-1}_{\beta\mu} \;\;,\;\; \mathcal{N}^\mu_\nu \;=\; \mathcal{U}_{\nu\alpha}(-\not{p} + m)_{\alpha\beta}\mathcal{U}^{-1}_{\beta\mu}$$

and

$$\mathcal{W}^{\mu\sigma}_{\nu\rho} \;=\; \mathcal{U}_{\nu\alpha}\mathcal{U}_{\rho\gamma}\, W^{\beta\delta}_{\alpha\gamma}\, \mathcal{U}^{-1}_{\beta\mu}\mathcal{U}^{-1}_{\delta\sigma}.$$

To diagonalize (4.31) it is necessary to introduce the similar transformation

$$\mathbf{O} \;=\; \exp\Big(-\, \mathcal{W}^{\mu\sigma}_{\nu\rho}\mathbf{a}^\nu \mathbf{b}_\mu \mathbf{a}^\rho \mathbf{b}_\sigma\Big). \qquad (4.32)$$

This transformation changes the operator $\mathbf{R}$ on the right hand side of (4.31) to

$$\mathbf{O}\;\mathbf{R}\;\mathbf{O}^{-1} \;=\; -8m + \mathcal{M}^\mu_\nu \mathbf{a}^\dagger_\nu \mathbf{a}^\mu + \mathcal{N}^\mu_\nu \mathbf{b}^{\dagger\nu}\mathbf{b}_\mu. \qquad (4.33)$$

The kets are transformed as

$$\mid \tilde{P}\rangle \;=\; \mathbf{O} \mid P\rangle.$$

The equation (4.33) has 2^8 different eigenstates with generally complex eigenvalues. The real parts of the complex eigenvalues are discreete equal to $-m, -2m, ... - 7m$, the imaginary parts are functions of the momentum p. The only two eigenstates with pure real eigenvalues 0 and $-8m$ are nondegenerate.

The eigenstate corresponding to the eigenvalue $\lambda_0 = 0$ can be written as

$$\mid \tilde{P}_0\rangle \;=\; \prod_{\nu=0}^{3} \mathbf{a}^\dagger_\nu \prod_{\mu=0}^{3} \mathbf{b}^{\dagger\mu} \mid 0\rangle. \qquad (4.34)$$

Thus any solution to (4.1) averaged over stochastic terms in this representation (and this order of approximation) can be written as linear superposition of eigenstates

$$\mid \langle\tilde{\rho}(t)\rangle\; \rangle \;=\; \sum_{i=0}^{255} e^{\lambda_i t}\, c_i \mid \tilde{P}_i\rangle \qquad (4.35)$$

and in the long time limit ($t \to \infty$) survives only the first term in the sum (4.35). Simultaneously vanishes the term $\mathcal{O}^{\mu\sigma}_{\nu\rho}(\exp[t(-2m \pm M)])$ in (4.27). To untangle the transformations done, it is necessary to calculate

$$\mid P_0\rangle \;=\; \mathbf{O^{-1}} \mid \tilde{P}_0\rangle \;=\; \exp\Big(\mathcal{W}^{\mu\sigma}_{\nu\rho}\mathbf{a}^\nu \mathbf{b}_\mu \mathbf{a}^\rho \mathbf{b}_\sigma\Big) \prod_{\alpha=0}^{3} \mathbf{a}^\dagger_\alpha \prod_{\beta=0}^{3} \mathbf{b}^{\dagger\beta} \mid 0\rangle. \qquad (4.36)$$

The result is

$$\mid P_0\rangle \;=\; f(p)\; \exp\Big[2\; (\mathcal{W}^{-1})^{\mu\sigma}_{\nu\rho}\mathbf{a}^{\dagger\nu}\mathbf{b}^\dagger_\mu \mathbf{a}^{\dagger\rho}\mathbf{b}^\dagger_\sigma\Big] \mid 0\rangle. \qquad (4.37)$$

where

$$f(p) \;=\; \left[\frac{r^2\,[4p^2+(2m\mp M)^2]^2 \;+\; (2m\mp M)^4}{m^2\,(2m\mp M)^2\;[p^2+(m\mp\frac{M}{2})^2]^2}\right] \tag{4.38}$$

The transformation back to nondiagonal $\not{p}$ matrices is straightforward.

Thus in the long time limit the expectation value of any functional $\langle \mathbf{F}[\psi,\bar\psi]\rangle$ is given by

$$\langle \mathbf{F}[\psi,\bar\psi]\rangle \;=\; \int \mathbf{F}[\psi,\bar\psi]\; P_0[\psi,\bar\psi]\; D\psi D\bar\psi \tag{4.39}$$

or

$$\langle \mathbf{F}[\psi,\bar\psi]\rangle \;=\; \frac{\int \mathbf{F}[\psi,\bar\psi]\;\exp(-r^{-1}\,\mathbf{S}[\psi,\bar\psi])\,D\psi D\bar\psi}{\int \exp(-r^{-1}\,\mathbf{S}[\psi,\bar\psi])\,D\psi D\bar\psi} \tag{4.40}$$

where

$$\mathbf{S}[\psi,\bar\psi] \;=\; \bar\psi_\nu\bar\psi_\rho\,(\,\frac{1}{2}\not{p}_{\nu\sigma}\not{p}_{\rho\mu} - [\frac{1}{2}p^2 + (2m\mp M)^2]\delta_{\nu\sigma}\delta_{\rho\mu})\psi_\mu\psi_\sigma\,. \tag{4.41}$$

5 Conclusions

Described method of the stochastic quantization of the fermion field yields as the expectation value of the combination $\bar\psi_\alpha\psi^\alpha\bar\psi_\beta\psi^\beta$ the scalar boson propagator. The procedure leading to this result is based on the concept of Markovian processes of Grassmann variables. The notion of derivative and integral in the Grassmann number calculus allows not only to introduce the concept of probability but also to extend it to stochastic processes. The explicit form of one particular process has been constructed. This process is analogical to classical Ohrenstein-Uhlenbeck process in the sense that all higher moments are products of the second moments. However, contrary to the classical process, in the case of Grassmann variables both negative and positive exponents in autocorrelation exponential are admissible. This sign appears in the resulting formulae as the sign of the term which can be interpreted as the mass defect. The negative mass defect stems from the process which has the "forbidden" sign.

Calculation of the field correlations proceeds according to standard stochastic quantization procedure. The new problem arises with the necessity to reduce the number of dimensions both of the stochastic time space and of the Euclidean space. This is connected with transition from a four point function to a two point function. The reduction of two degrees of freedom in stochastic time follows quite naturally from the properties of the probability distribution of the stochastic terms. The manifold where it is nonzero is twodimensional. Staying on this manifold means, that the product $\bar\psi_\alpha\psi^\alpha$ is always taken in the same stochastic time instant, which is natural if one wants to interpret it as one particle composed of ψ and $\bar\psi$.

The initial system consisted of four fermions and the final system represents a boson field. Thus the coordinates of individual constituents disappear from description of the final system. However, the stochastic quantization provides no mechanism to do this. Hence the redundant degrees of freedom in Euclidean space are reduced by an ad hoc procedure which is based on transformation of the momenta to the center of mass system of $\bar\psi_\alpha\psi^\alpha$ and its conjugate (which is identical in this case). The whole expression $\langle\bar\psi_\alpha\psi^\alpha\bar\psi_\beta\psi^\beta\rangle$ is then integrated over space variable corresponding to the momentum of the overall center of mass p^*. Consequently p^* appears as an argument of a δ-function and it is set to zero

by integration. The whole system is not considered to be on mass shell. Two other space variables v and w (3.18) are also set to zero which has the meaning of putting the pair ψ and $\bar{\psi}$ into one space point x resp. y. As a result the final boson propagator depends only on the distance of two points in Euclidean space and the systems $\langle .. \rangle$ and $\overline{\langle .. \rangle}$ propagate with opposite momenta. The energy-momentum conservation constraint enters the whole description only in this sense.

The dynamics of the fermion field affected by stochastic process (2.18,2.19) is described in the long time limit by (4.41). It consists of the part which is up to the coefficient of the mass term equal to free scalar boson Lagrangian and the term which is analog to the scalar field term in Stueckelberg's Lagrangian of a massive vector boson field. The functional (4.39) does not act on any arbitrary scalar function ϕ but only on spinor functions. Hence this term will lead in case of a scalar constructed from spinor functions ($\mathbf{F}[\bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta]$) to $Tr(\not{p}\not{p})$ and (4.41) will turn to free scalar boson Lagrangian with no selfinteraction. In case of other functions constructed from spinors, (4.40) gives generally nonzero results - e.g. $\mathbf{F}[\bar{\psi}\gamma^\mu\psi\,\bar{\psi}\gamma^\nu\psi]$ results in constant $-\delta^{\mu\nu}/m(2m \mp M)$. However, it has to be stressed that the probability distribution of stochastic terms is just the most simple one satisfying conditions of a Markovian process with no ambition to describe any particular dynamics in the long time limit.

The long time behaviour of the whole system is studied starting from the expansion of the solution to the Liuville equation. Thus the quality of approximation by Fokker-Planck equation can be controlled. The convergence of the expansion is given by numerical values of r and fermion mass m as the coefficients corresponding to $m\,(2m \mp M)$ in denominator of S_1 (4.20) become equal to $m^2\,(6m \mp M)(2m \mp M)$, $m^3\,(10m \mp M)(6m \mp M)(2m \mp M)$, ..etc in higher terms S_2, S_3, ... The Gaussian behaviour of stochastic terms implies that higher order terms should reduce to products of free "boson Lagrangians" (each containing the term discussed above) with masses increasing by $4m$. However, then the procedure leading to reduction of the degrees of freedom has to be generalized and the result will depend on it.

The whole calculation closely follows the standard stochastic quantization procedure. The only two differences in the approach consist in different statistical properties of the stochastic terms and in reduction of the degrees of freedom in final formulae. The later difference is clearly connected with desired "coarse graining" of the description. On the other hand the statistical properties of stochastic terms in Langevin equations are responsible for the fact that in the long (stochastic) time limit the free fermions disappear and the combination $\langle \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \rangle$ propagates as a scalar boson field. Thus the stochastic terms can be regarded as the external field coupling the fermions together. The question whether this interaction is an effective result of a standard interaction via gauge fields lays beyond the scope of the described procedure as the statistical nature of stochastic terms is imposed ad hoc. Any hint that the given method reflects real dynamics would generate a puzzling problem - mere statistical properties of stochastic term make difference between standard stochastic quantization of free fermion field and description of interacting system which gets quantized "along the way" in course of calculation.

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